

# SARGE: an algorithm for generating QCD-antennas

André van Hameren\* and Ronald Kleiss†

University of Nijmegen, Nijmegen, the Netherlands

Petros D. Draggiotis‡

University of Nijmegen, Nijmegen, the Netherlands

Institute of Nuclear Physics, NCSR Δημόκριτος, Athens, Greece

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## Abstract

We present an algorithm to generate any number of random massless momenta in phase space, with a distribution that contains the kinematical pole structure that is typically found in multi-parton QCD-processes. As an application, we calculate the cross-section of some  $e^+e^- \rightarrow$  partons processes, and compare **SARGE**'s performance with that of the uniform-phase space generator **RAMBO**.

Considering that many multi-jet processes will occur in future hadron colliders, such as the LHC, it is necessary to calculate their cross-sections. A part of the amplitude of these processes consists of a multi-parton QCD-amplitude, and it is well known [1] that the leading kinematic singularity structure of the squared matrix elements is given by the so-called *antenna pole structure* (APS). In particular, for  $n$  gluons it is given by all permutations in the momenta of

$$\frac{1}{(p_1 p_2)(p_2 p_3)(p_3 p_4) \cdots (p_{n-1} p_n)(p_n p_1)} \quad , \quad (1)$$

where  $(p_i p_j)$  denotes the Lorentz invariant scalar product of the gluon momenta  $p_i$  and  $p_j$ . Actually, it is this kinematical structure that is implemented in algorithms based on the so called **SPHEL** approximation to calculate the amplitudes [1]. But it is expected, and observed, that the same structure occurs in the exact matrix elements [2, 3].

For the integration of the differential cross-sections of the processes under consideration, the Monte Carlo method is the only option, and a phase space generator is needed. **RAMBO** [4] is a robust and efficient algorithm to generate any number of random massless momenta in their center-of-mass frame (CMF) with a given energy. However, **RAMBO** generates the momenta distributed uniformly in phase space, so that a large number of events is needed to integrate integrands with the APS to acceptable precision. Especially when the evaluation of the integrand is time-consuming, which is the case for the exact matrix elements, this is highly inconvenient.

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\*andrevh@sci.kun.nl, †kleiss@sci.kun.nl, ‡petros@sci.kun.nl

In this paper, we introduce **SARGE**, an algorithm to generate any number of random massless momenta in their CMF with a given energy, distributed with a density that contains the APS. We shall show that it takes account for a substantial reduction in computing time in the calculation of cross-sections of multi-parton processes. We briefly sketch the outline of the **SARGE**-algorithm; a fuller discussion, appropriate to hadronic initial states as well, will be given elsewhere [7].

The name **SARGE** stands for Staggered Antenna Radiation GEnerator, and is inspired by the structure of the algorithm. It consists of the repeated use of the *basic antenna* density for the generation of a momentum  $k$ , given two momenta  $p_1$  and  $p_2$ :

$$dA(p_1, p_2; k) = d^4k \delta(k^2) \theta(k^0) \frac{1}{\pi} \frac{(p_1 p_2)}{(p_1 k)(k p_2)} g\left(\frac{(p_1 k)}{(p_1 p_2)}\right) g\left(\frac{(k p_2)}{(p_1 p_2)}\right) . \quad (2)$$

Here,  $g$  is a function that serves to regularize the infrared and collinear singularities, as well as to ensure normalization over the whole space for  $k$ : therefore,  $g(\xi)$  has to vanish sufficiently fast for both  $\xi \rightarrow 0$  and  $\xi \rightarrow \infty$ . At this point, we take the simplest possible function we can think of, that has a sufficiently regularizing behavior. We introduce a positive non-zero number  $\xi_m$  and take

$$g(\xi) = \frac{1}{2 \log \xi_m} \theta(\xi - \xi_m^{-1}) \theta(\xi_m - \xi) , \quad (3)$$

which forces the value of  $\xi$  to be between  $\xi_m^{-1}$  and  $\xi_m$ , and is normalized such that  $\int dA = 1$ . Let us immediately adopt the notation

$$\xi_1 = \frac{(p_1 k)}{(p_1 p_2)} \quad \text{and} \quad \xi_2 = \frac{(k p_2)}{(p_1 p_2)} . \quad (4)$$

The main motivation to make the regularizing function depend on  $\xi_1$  and  $\xi_2$  is that it makes  $dA$  completely invariant under Lorentz- and scale transformations of the momenta. Consequently, the number  $\xi_m$  gives a cut-off for the quotients  $\xi_1$  and  $\xi_2$  of the scalar products of the momenta, and not for the scalar products themselves. It is, however, possible to relate  $\xi_m$  to the total energy  $\sqrt{s}$  in the CMF and a cut-off  $s_0$  on the invariant masses, i.e., the requirement that

$$(p_i + p_j)^2 \geq s_0 \quad (5)$$

for all pairs of momenta  $p_i \neq p_j$ . This can be done by choosing

$$\xi_m = \frac{s}{s_0} - \frac{(n+1)(n-2)}{2} , \quad (6)$$

where  $n$  is the total number of momenta. With this choice, the invariant masses  $(p_1 + k)^2$  and  $(k + p_2)^2$  are regularized, but can still be smaller than  $s_0$  so that the whole of the demanded phase space is covered. The  $s_0$  can be derived from physical cuts  $p_T$  on the transverse momenta and  $\theta_0$  on the angles between the outgoing momenta:

$$s_0 = 2p_T^2 \cdot \min \left( 1 - \cos \theta_0 , \left( 1 + \sqrt{1 - p_T^2/s} \right)^{-1} \right) . \quad (7)$$

We now give the algorithm to generate  $k$  under the basic antenna density. Let  $k^0$ ,  $\phi$  and  $\theta$  denote the absolute value, the polar angel and the azimuthal angle of  $\vec{k}$  in the frame for which  $\vec{p}_1 = -\vec{p}_2$  with  $\vec{p}_1$  along the positive  $z$ -axis. To generate  $k$ , one should

**Algorithm 1 (BASIC ANTENNA)**

1. determine the direction of  $\vec{p}_1$  in the CMF of  $p_1$  and  $p_2$ ;
2. generate two numbers  $\xi_1$ ,  $\xi_2$  independently, each from the density  $g(\xi)/\xi$ ;
3. compute from these the values  $k^0$  and  $\cos \theta$ ;
4. generate  $\phi$  uniformly in  $[0, 2\pi)$ ;
5. construct the momentum  $k$  in the CMF of  $p_1$  and  $p_2$ ;
6. boost the result to the actual frame in which  $p_1$  and  $p_2$  were given.

The **RAMBO** algorithm was developed with the aim to generate the flat phase space distribution of  $n$  massless momenta as uniformly as possible. The differential density is given by

$$dV_n(\{p\}) = \delta(\sqrt{s} - P^0) \delta^3(\vec{P}) \prod_{i=1}^n d^4 p_i \delta(p_i^2) \theta(p_i^0) , \quad (8)$$

where  $P = \sum_{i=1}^n p_i$ . Let us denote

$$dA_{j,k}^i = dA(q_j, q_k; q_i) , \quad \text{and} \quad \xi_{k,l}^{i,j} = \frac{(p_i p_j)}{(p_k p_l)} . \quad (9)$$

To include the APS in the density, one should

**Algorithm 2 (QCD ANTENNA)**

1. generate massless momenta  $q_1$  and  $q_n$  in CMF;
2. generate  $n - 2$  momenta  $q_j$  by the basic antennas  $dA_{1,n}^2 dA_{2,n}^3 dA_{3,n}^4 \cdots dA_{n-2,n}^{n-1}$ ;
3. compute  $Q = \sum_{j=1}^n q_j$ , and the boost and scaling transforms that bring  $Q^0$  to  $\sqrt{s}$  and  $\vec{Q}$  to  $(0, 0, 0)$ ;
4. for  $j = 1, \dots, n$ , boost and scale the  $q_j$  accordingly, into the  $p_j$ .

This way, the momenta  $p_j$  are generated with differential density  $dV_n(\{p\}) A_n^{\text{QCD}}(\{p\})$ , where

$$A_n^{\text{QCD}}(\{p\}) = \frac{s^2}{2\pi^{n-1}} \cdot \frac{g(\xi_{1,n}^{1,2}) g(\xi_{1,n}^{2,n}) g(\xi_{2,n}^{2,3}) g(\xi_{2,n}^{3,n}) \cdots g(\xi_{n-2,n}^{n-2,n-1}) g(\xi_{n-2,n}^{n-1,n})}{(p_1 p_2)(p_2 p_3)(p_3 p_4) \cdots (p_{n-1} p_n)(p_n p_1)} . \quad (10)$$

We point out that, whereas the product  $dA_{1,n}^2 \cdots dA_{n-2,n}^{n-1}$  contains a factor  $(p_1 p_n)$  in the numerator, the scaling transformation carries a Jacobian that is precisely  $s^2/(p_1 p_n)^2$ , thus leading to a perfectly symmetric APS.

Usually, the event generator is used to generate cut phase space. If a generated event does not satisfy the physical cuts, it is rejected. In the calculation of the weight coming with an event, the only contribution coming from the functions  $g$  is, therefore, their normalization. In total, this gives a factor  $1/(2 \log \xi_m)^{2n-4}$  in the density.

Because we are dealing with gluon momenta, we want to symmetrize the density. This can be done by re-labeling the momenta using a random permutation:

**Algorithm 3 (SYMMETRIZATION)**

1. generate a random permutation  $\sigma \in S_n$  and put  $p_i \leftarrow p_{\sigma(i)}$  for all  $i = 1, \dots, n$ .

An algorithm to generate the random permutations can be found in [5]. As a result, the differential density becomes

$$dV_n(\{p\}) \left( \frac{1}{n!} \sum_{\text{perm.}} A_n^{\text{QCD}}(\{p\}) \right) , \quad (11)$$

where the sum is over all permutations of  $(1, \dots, n)$ . An efficient algorithm to calculate a sum over permutations can be found in [1].

When doing calculations with this algorithm on a phase space cut such that  $(p_i + p_j)^2 > s_0$  for all  $i \neq j$  and some reasonable  $s_0 > 0$ , we notice that a very high percentage of the generated events does not pass the cuts. An important reason why this happens is that the cuts, generated by the choices of  $g$  (Eq. (3)) and  $\xi_m$  (Eq. (6)), are implemented only on the variables  $\xi_{k,l}^{i,j}$  that appear explicitly in the generation of the QCD-antenna. Therefore, an improvement is obtained as follows. Let  $\mathbf{P}_m$  denote the subspace of  $[-1, 1]^m$  for which  $|x_i - x_j| \leq 1$  for all  $i, j = 1, \dots, m$ , and let us denote the number of  $\xi_{k,l}^{i,j}$ -variables that has to be generated  $n_\xi = 2n - 4$ . An improvement is obtained if the generation of these variables is replaced by

**Algorithm 4 (IMPROVEMENT)**

1. generate  $(x_1, \dots, x_{n_\xi})$  distributed uniformly in  $\mathbf{P}_{n_\xi}$ ;
2. define  $x_0 = 0$  and put, for all  $i = 2, \dots, n - 1$ ,

$$\xi_{i-1,n}^{i-1,i} \leftarrow e^{(x_{2i-3} - x_{2i-4}) \log \xi_m} , \quad \xi_{i-1,n}^{i,n} \leftarrow e^{(x_{2i-2} - x_{2i-4}) \log \xi_m} . \quad (12)$$

Because all the variables  $x_i$  are distributed uniformly such that  $|x_i - x_j| \leq 1$ , all quotients  $\xi_{k,l}^{i,j}$  with  $(i, j)$  and  $(k, l)$  in  $\{(i-1, i), (i, n) \mid i = 2, \dots, n-1\}$  are distributed such that they satisfy  $\xi_m^{-1} \leq \xi_{k,l}^{i,j} \leq \xi_m$ . This is an improvement on the previous situation, because then only the quotients  $\xi_{i-1,n}^{i-1,1}$  and  $\xi_{i-1,n}^{i,n}$  with  $i = 2, \dots, n-1$  satisfied the relation. In terms of the variables  $x_i$ , this means that the volume of  $\mathbf{P}_{n_\xi}$  is generated, which is  $n_\xi + 1$ , instead of the volume of  $[-1, 1]^{n_\xi}$ , which is  $2^{n_\xi}$ . We have to note here that this improvement only makes sense because there is a very efficient algorithm to generate the uniform distribution in  $\mathbf{P}_m$  [6]. The total density changes such that the product of the  $g$ -functions in Eq. (10) has to be replaced by

$$g_{n-2}^{\mathbf{P}}(\xi_m; \{\xi\}) = \frac{1}{(n_\xi + 1)(\log \xi_m)^{n_\xi}} \times \begin{cases} 1 & \text{if } (x_1, \dots, x_{n_\xi}) \in \mathbf{P}_{n_\xi} , \\ 0 & \text{if } (x_1, \dots, x_{n_\xi}) \notin \mathbf{P}_{n_\xi} , \end{cases} \quad (13)$$

where the variables  $x_i$  are functions of the variables  $\xi_{k,l}^{i,j}$  as defined by (12). Again, only the normalization has to be calculated for the weight of an event.

We compare **SARGE** with **RAMBO** in the calculation of the cross-section of the processes

$$e^+e^- \longrightarrow \gamma^* \longrightarrow q\bar{q}g, q\bar{q}q\bar{q}, q\bar{q}q'\bar{q}', q\bar{q}gg, q\bar{q}q\bar{q}g, q\bar{q}q'\bar{q}'g, q\bar{q}ggg \quad . \quad (14)$$

The squared matrix element was calculated with the algorithm presented in [2], suitably adapted for these processes. We used massless electrons and quarks, and took the sum over final-state helicities and the average over initial-state helicities. We also summed over the color configurations of the final states. The center-of-mass energy  $\sqrt{s}$  was fixed to 500 GeV for the processes with 5 outgoing momenta, and to 100 GeV for the other processes. The cuts on the phase space were fixed with choices of a parameter  $\tau$ , which is related to the cut-off  $s_0$  on the squares of the outgoing momenta (Eq. (5)) by

$$s_0 = \frac{2s\tau}{n(n-1)} \quad , \quad (15)$$

where  $n$  is the number of outgoing momenta. If  $\tau = 1$ , then  $s_0$  is larger than the maximal value that is kinematically allowed. The couplings and charges in various processes were all set to the value 1, since they only contribute a factor to the cross-section, which is irrelevant for this analysis. The results of the computer runs are given in the tables below. Presented are the final result for the cross-section  $\sigma$  in units of  $\text{GeV}^{-2}$ , the number of generated events  $N_{\text{ge}}$ , the number of accepted events  $N_{\text{ac}}$ , and the cpu-time consumed  $t_{\text{cpu}}$  in seconds. All Monte Carlo runs were performed on a single 440-MHz UltraSPARC-IIi processor, and were stopped when an expected error of 3% was reached.

The final results for the cross-sections are irrelevant in our discussion, and are just printed to show that the results with **SARGE** and **RAMBO** are compatible within the 3% error estimate. The most important conclusion that can be drawn from the results is that **SARGE** needs less accepted events than **RAMBO** for the given error estimate, especially for small values of  $\tau$ , *i.e.*, for phase space that comes close to the singularities of the QCD-amplitudes. (Remember that the ratio of the volumes of cut phase space and whole phase space is given by  $N_{\text{ac}}/N_{\text{ge}}$  for **RAMBO**.) As a result, less evaluations of the matrix elements have to be done which accounts for a large gain in computer time. It is true that **SARGE** is “ineffective” in the sense that many of the generated events have to be rejected because they do not satisfy the cuts imposed, but this is fully compensated by the fact that generating random numbers is much cheaper than evaluating matrix elements nowadays. For the last four processes, no results with **RAMBO** and  $\tau = 0.01$  are presented, but we observe that  $t_{\text{cpu}} > 130,000$  seconds. The fraction of phase space covered with five massless momenta and  $\tau = 0.01$  is  $0.893 \pm 0.001$ .

$e^+e^- \rightarrow q\bar{q}g$								
$\tau$	0.5		0.1		0.05		0.01	
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO
$\sigma$	1.85e-5	1.85e-5	1.53e-4	1.58e-4	2.61e-4	2.66e-4	6.26e-4	6.41e-4
$N_{\text{ge}}$	7,691	25,782	10,777	24,801	10,806	37,121	11,437	366,614
$N_{\text{ac}}$	5,503	6,536	9,436	20,112	9,852	33,577	10,860	359,447
$t_{\text{cpu}}$	251	293	429	899	451	1,503	497	16,124

$e^+e^- \rightarrow q\bar{q}q\bar{q}$								
$\tau$	0.5		0.1		0.05		0.01	
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO
$\sigma$	9.79e-9	10.4e-9	7.72e-7	7.86e-7	1.90e-6	1.83e-6	7.39e-6	7.00e-6
$N_{\text{ge}}$	64,384	158,678	32,492	27,091	34,701	29,642	41,744	113,368
$N_{\text{ac}}$	4,428	4,551	9,894	15,328	13,081	22,297	20,150	107,021
$t_{\text{cpu}}$	775	786	1,718	2,606	2,256	3,778	3,578	18,038

$e^+e^- \rightarrow q\bar{q}q'\bar{q}'$								
$\tau$	0.5		0.1		0.05		0.01	
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO
$\sigma$	5.38e-9	5.30e-9	4.07e-7	4.24e-7	1.00e-6	1.02e-6	3.95e-6	3.89e-6
$N_{\text{ge}}$	98,840	245,138	50,052	45,963	63,398	50,873	71,254	366,166
$N_{\text{ac}}$	6,696	7,022	15,392	25,883	23,989	38,145	34,584	345,323
$t_{\text{cpu}}$	1,165	1,198	2,664	4,346	4,133	6,434	5,843	58,708

$e^+e^- \rightarrow q\bar{q}gg$							
$\tau$	0.5		0.1		0.05		0.01
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE
$\sigma$	1.76e-7	1.70e-7	1.86e-5	1.95e-5	5.19e-5	5.27e-5	5.40e-4
$N_{\text{ge}}$	96,942	268,407	42,321	86,608	50,552	298,073	50,414
$N_{\text{ac}}$	6,579	7,677	12,945	48,902	19,091	223,530	26,551
$t_{\text{cpu}}$	1,363	1,597	3,619	6,398	3,802	43,913	5,287

$e^+e^- \rightarrow q\bar{q}q\bar{q}g$							
$\tau$	0.5		0.1		0.05		0.01
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE
$\sigma$	2.04e-11	1.91e-11	4.05e-8	4.08e-8	1.68e-7	1.61e-7	1.48e-6
$N_{\text{ge}}$	4,028,648	4,017,888	238,220	97,035	203,237	210,325	176,710
$N_{\text{ac}}$	5,616	5,094	14,216	33,239	19,522	121,734	29,492
$t_{\text{cpu}}$	4,530	3,941	10,333	23,875	14,159	87,756	21,407

$e^+e^- \rightarrow q\bar{q}q'\bar{q}'g$							
$\tau$	0.5		0.1		0.05		0.01
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE
$\sigma$	1.05e-11	1.05e-11	2.19e-8	2.23e-8	9.07e-8	8.86e-8	7.85e-7
$N_{\text{ge}}$	5,596,725	6,929,475	436,225	188,693	377,384	522,602	305,426
$N_{\text{ac}}$	7,730	8,844	26,154	64,558	36,042	302,724	51,044
$t_{\text{cpu}}$	5,882	6,494	17,595	43,104	24,764	201,801	34,700

$e^+e^- \rightarrow q\bar{q}ggg$							
$\tau$	0.5		0.1		0.05		0.01
alg.	SARGE	RAMBO	SARGE	RAMBO	SARGE	RAMBO	SARGE
$\sigma$	1.31e-11	1.30e-11	3.63e-7	3.54e-7	1.63e-6	1.54e-6	1.85e-5
$N_{\text{ge}}$	5,926,016	6,650,538	366,538	131,617	303,003	186,257	335,307
$N_{\text{ac}}$	8,194	8,475	21,918	45,157	29,018	107,897	56,008
$t_{\text{cpu}}$	7,407	7,398	18,120	36,958	24,036	88,318	46,673

As an extra illustration, we also present the convergence to zero of the expected error during the Monte Carlo-run for a few cases. In Fig. 1, we plot the relative error as function of the number of generated events using a double-log scale. We first of all observe that the curves for **SARGE** are less spiky, which shows that **SARGE** takes care for a substantial part of the singular behavior of the integrand. Every time a **RAMBO**-event hits a singularity, a term much larger than the average so far is added to the Monte Carlo sum, resulting in an increase of the expected error. Furthermore, we observe that the **SARGE**-error converges quicker than the **RAMBO**-error, except in the case of  $e^+e^- \rightarrow q\bar{q}ggg$  with  $\tau = 0.05$ . However, this is a plot of the error as function of the number of generated events, and we know that many **SARGE**-events have to be rejected. A more realistic view is given by a plot of the error as function of cpu-time (Fig. 2), which clearly shows that **SARGE** outperforms **RAMBO**.

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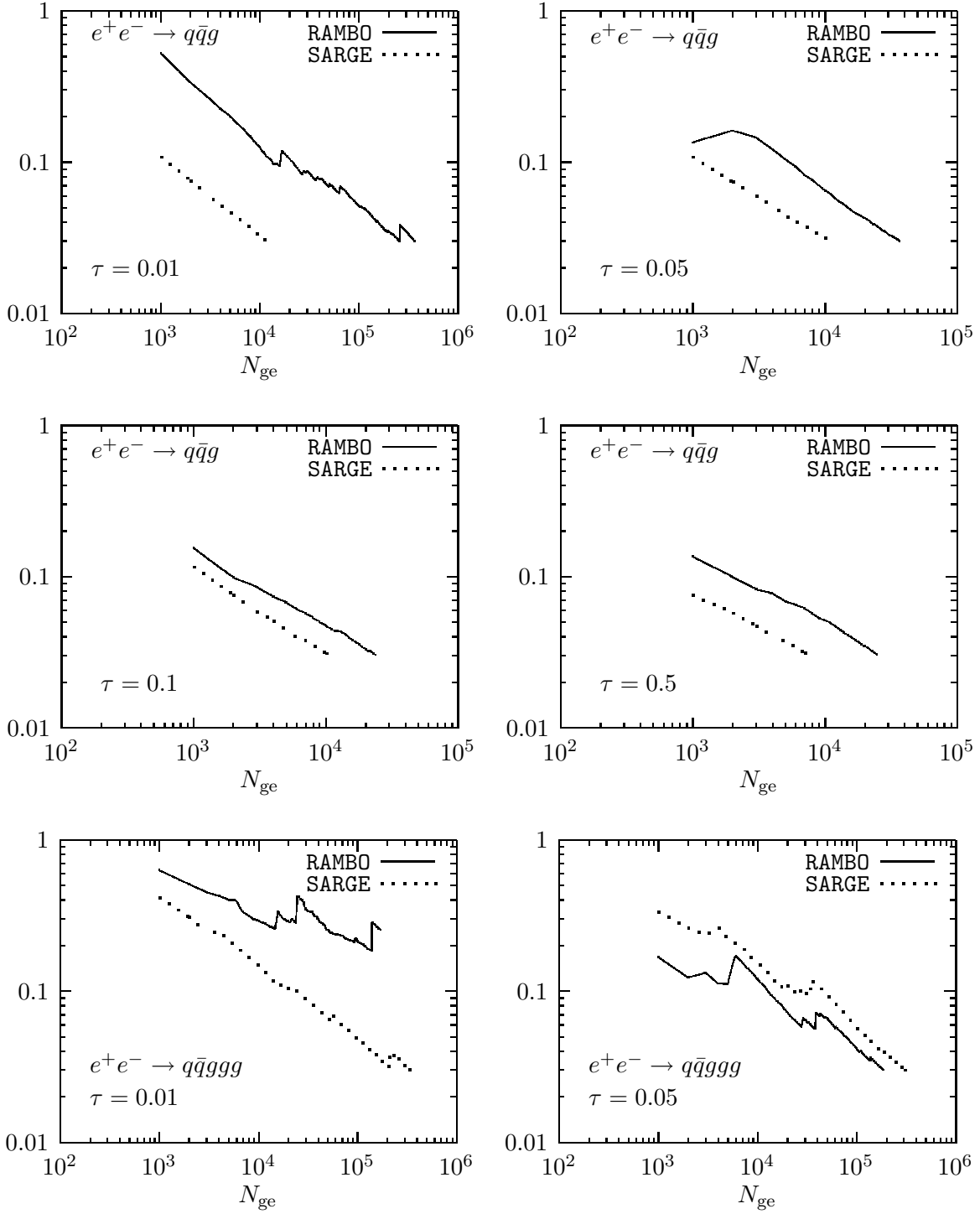


Figure 1: The expected relative error as function of the number of generated events.



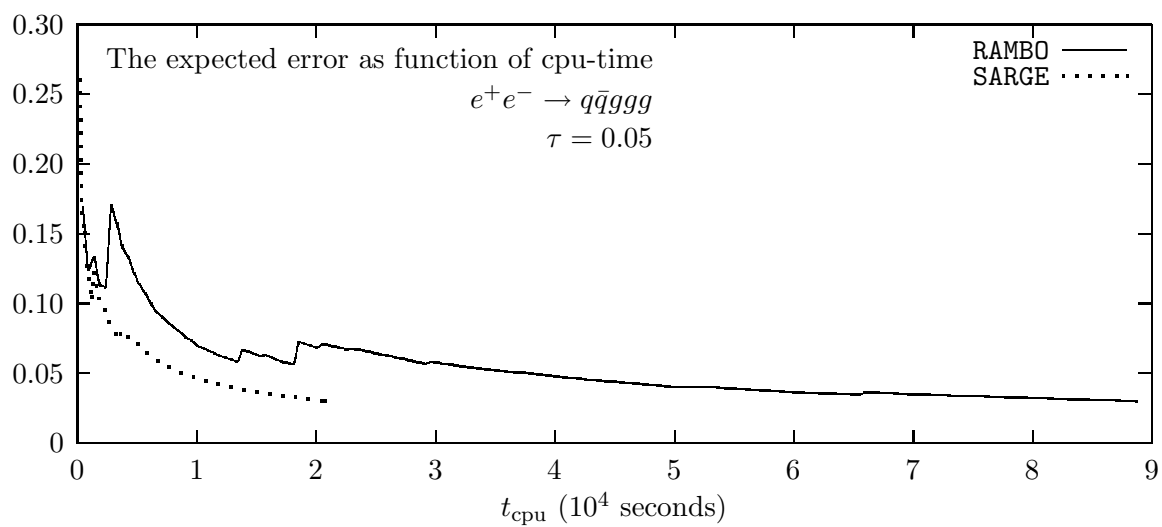


Figure 2: The expected relative error as function of cpu-time.